

ON THE BEHAVIOR OF THE SIZE OF A MONOMIAL IDEAL

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ABSTRACT. In this paper we study the behavior of the size of a monomial ideal under polarization and under generic deformations. As an application, we extend a result relating the size and the Stanley depth of a squarefree monomial ideal obtained by Herzog, Popescu and Vlodoiu, together with a parallel result obtained by Tang.

1. INTRODUCTION

Let $S = \mathbb{K}[X_1, \dots, X_n]$, with \mathbb{K} a field, and let $I \subset S$ be a monomial ideal. The notion of *size* of a monomial ideal was introduced by Lyubeznik in [12]. In time, it has been used by several authors, see for example [8], [9], [15], [16] and [17].

Several algebraic or combinatorial invariants associated to a monomial ideal are known to have a nice behavior under *polarization*. For example, see [3], [5] or [10]. In the first part of this paper we study the behavior of the size I under polarization. In Section 3 we establish that

$$\text{size } I^p \leq \text{size } I + c,$$

where $I^p \subset S' = \mathbb{K}[X_1, \dots, X_{n'}]$ is the polarization of I and $c = n' - n$ (see Theorem 3.7). The equality does not hold in general, as shown in Example 3.9.

In the main result of this paper, that is Theorem 3.10, we provide a complete description of the (particular) situation when the equality

$$\text{size } I^p = \text{size } I + c$$

does hold.

A counterexample by H. Shen shows that the second statement of [8, Lemma 3.2] is false when I is not squarefree. It follows that the proof [8, Theorem 3.1] is correct only when I is squarefree, and that the statement of [8, Theorem 3.1] is in fact a conjecture in general. As an application of our main result we deduce in Corollary 3.15 that this conjecture is true under the conditions described in Theorem 3.10. In the same Corollary and under the same conditions we also obtain an extension of [17, Theorem 3.2].

The notion of *deformation* of a monomial ideal was introduced by Bayer et al. [2] and further developed in Miller et al. [13]. The most important deformations are the generic deformations, which attracted the attention of several researchers, see for example [1] or [11]. In the last part of this paper (Section 4) we briefly study the behavior of the size I under generic deformations. We find that

$$\text{size } I_\varepsilon \leq \text{size } I,$$

where I_ε is a generic deformation of I (see Proposition 4.4).

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2. PREREQUISITES

Let $S = \mathbb{K}[X_1, \dots, X_n]$, with \mathbb{K} a field. For $n \in \mathbb{N}$ we use the notation $[n] := \{1, \dots, n\}$.

2.1. The size of an ideal. In this Subsection we recall the definition of size and we make some easy remarks; these will be needed in the sequel.

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^s U_i$ an irredundant primary decomposition of I , with U_i monomial ideals. Let U_i be V_i -primary. Then each V_i is a monomial prime ideal and $\text{Ass}(S/I) = \{V_1, \dots, V_s\}$.

Definition 2.1. Following Lyubeznik [12, Proposition 2] the *size* of I , denoted in the following by $\text{size} I$, is the number $v + (n - h) - 1$, where v is the minimum number $t \leq s$ such that there exist $i_1 < \dots < i_t$ with

$$\sqrt{\sum_{j=1}^t U_{i_j}} = \sqrt{\sum_{i=1}^s U_i},$$

and where $h = \text{height} \sum_{i=1}^s U_i$.

A monomial ideal is *irreducible* if it is generated by powers of some variables. An irreducible decomposition of a monomial ideal I is an expression $I = I_1 \cap \dots \cap I_r$ with the I_j irreducible. Every irreducible monomial ideal is primary, so an irreducible decomposition is a primary decomposition [7, Theorem 1.3.1].

Remark 2.2. Observe that $\sqrt{\sum_{j=1}^t U_{i_j}} = \sum_{j=1}^t V_{i_j}$ and $\sqrt{\sum_{i=1}^s U_i} = \sum_{i=1}^s V_i$, so that the size of I only depends on the set of associated prime ideals of S/I . Indeed, consider an irredundant irreducible decomposition

$$I = \bigcap_{l=1}^r Q_l, \quad r \leq s,$$

where Q_l are monomial ideals and where $P_l = \sqrt{Q_l}$. Then, for each $i \in [s]$ we have that

$U_i = \bigcap_{j=1}^{k_i} Q_{l_j}$ where $P_{l_j} = V_i$. Using the definition of $\text{size} I$ we get the same result as above.

From now on we will only consider irredundant irreducible decompositions.

Remark 2.3. Working with size we can assume that $\sum_{i=1}^r P_i = \mathfrak{m}$, where \mathfrak{m} is the maximal ideal. Otherwise, set $X = \{X_1, \dots, X_n\}$, $Z = \{X_k \mid X_k \notin \sum_{i=1}^s P_i\}$, $T = \mathbb{K}[X \setminus Z]$ and let $J = I \cap T$. Then the sum of the associated prime ideals of J is the maximal ideal of T and

$$\text{size} I = \text{size} J + |Z|.$$

Remark 2.4. Let $I \subset S$ be a monomial ideal and let $I = \bigcap_{i=1}^r Q_i$ be an irredundant decomposition of I as an intersection of irreducible ideals, where $\sqrt{Q_i} = P_i$. Then $\sqrt{I} = \bigcap_{i=1}^r P_i$

and so we have that $\text{size} I = \text{size} \sqrt{I}$. In general it is easy to see that, if $\text{Ass} S/I \subseteq \text{Ass} S/J$ for two monomial ideals, then $\text{size} I \geq \text{size} J$.

2.2. The polarization of a monomial ideal. In the following we study the behavior of the size of a monomial ideal under polarization. We recall the definition of polarization following Herzog and Hibi [7]. Let $I \subset S$ be a monomial ideal with generators u_1, \dots, u_m , where $u_i = \prod_{j=1}^n X_j^{a_{ij}}$ for $i = 1, \dots, m$. For each j let $a_j = \max\{a_{ij} : i = 1, \dots, m\}$. Set $a = (a_1, \dots, a_n)$ and S' to be the polynomial ring

$$S' := \mathbb{K}[X_{k,l} : 1 \leq k \leq n, 1 \leq l \leq a_j].$$

Then the *polarization* of I is the squarefree monomial ideal $I^p \subset S'$ generated by v_1, \dots, v_m , where

$$v_i = \prod_{k=1}^n \prod_{l=1}^{a_{ij}} X_{k,l} \quad \text{for } i = 1, \dots, m.$$

The *Stanley depth* of an S -module M is a combinatorial invariant denoted in the following by $\text{sdepth} M$. We skip the details since this invariant will only appear briefly in Corollary 3.15. For an excellent account on the subject, the reader is referred to Herzog's survey [6]. The following Theorem follows immediately from the main result of [10].

Theorem 2.5. *Let $I \subset S = \mathbb{K}[X_1, \dots, X_n]$ be a monomial ideal and $I^p \subset S' = \mathbb{K}[X_1, \dots, X_{n'}]$ be the polarization of I . Set $c = n' - n$. Then*

- (1) $\text{sdepth} I^p = \text{sdepth} I + c$;
- (2) $\text{sdepth} S^p / I^p = \text{sdepth} S / I + c$.

Finally, we recall the most important known results relating sdepth and size .

Theorem 2.6. *Let I be a squarefree monomial ideal of S . Then*

- (1) $\text{sdepth} I \geq \text{size} I + 1$ (see [8, Theorem 3.1]);
- (2) $\text{sdepth} S / I \geq \text{size} I$ (see [17, Theorem 3.2]).

For an extension of Theorem 2.6 see [4].

3. THE BEHAVIOR OF SIZE UNDER POLARIZATION

Let $S = \mathbb{K}[X_1, \dots, X_n]$, with \mathbb{K} a field, and $I \subset S$ be a monomial ideal. Throughout the section, we fix $I = \bigcap_{i=1}^r Q_i$ to be the unique irredundant irreducible decomposition of I . For each $i \in [r]$ we have

$$Q_i = (X_1^{a_1^i}, \dots, X_n^{a_n^i}),$$

where $a_k^i \in \mathbb{N}$ for all $k \in [n]$. In this writing we use the following convention: Assume Q_i to be P_i -primary. Then, if $X_k \notin P_i$, we set $a_k^i = 0$ and $X_k^{a_k^i} = 0$.

Let $a_k = \max\{a_k^1, \dots, a_k^r\}$, for all $k \in [n]$. Denote by $n' = \sum_{k=1}^n a_k$ and set $c = n' - n$. Set

$$S' := \mathbb{K}[X_{k,l} : 1 \leq k \leq n, 1 \leq l \leq a_k].$$

Let $I^p \subset S'$ be the polarization of I and $Q_i^p \subset S'$ be the polarization of Q_i . Then

$$I^p = \bigcap_{i=1}^r Q_i^p$$

by [3, Proposition 2.3]. Moreover, by [3, Proposition 2.5], it holds

$$Q_i^p = \bigcap_{\substack{1 \leq b_k \leq a_k^i \\ 1 \leq k \leq n}} (X_{1,b_1}, \dots, X_{n,b_n})$$

with the convention that, if $a_k^i = 0$, then $b_k = X_{k,0} = 0$.

Definition 3.1. Remark that for all $i \in [r]$ there exists at least one index $k \in [n]$ such that $a_k^i \geq a_k^j$ for all $j \in [r]$. We say that a_k^i is a *top power* for Q_i . For a subset $N \subseteq [n]$ we set $B_N^i = \{a_k^i : a_k^i \text{ is a top power for } Q_i \text{ and } k \in N\}$. Note that $B_{[n]}^i \neq \emptyset$ for all $i \in [r]$. For a subset $R \subseteq [r]$ we introduce the notion *top base* (denoted by C by the following recursive algorithm). Let $M \in \mathcal{M}_{r,n}(\mathbb{Z})$ be the matrix with $M_{ij} = a_j^i$, for all $i \in [r]$, $j \in [n]$.

Algorithm 1: Function which computes a top base

Data: $r, n \in \mathbb{N}$, a Matrix $M \in \mathcal{M}_{r,n}(\mathbb{Z})$ and the Sets $R \subseteq [r]$, $N \subseteq [n]$

Result: A List C containing a top base

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1 Vector  $C = \text{NewVector}(r, 0)$ ,  $R = [r]$ ,  $N = [n]$ ;
2 List BuildTopBase( $M, R, N$ );
begin
3   if  $R = \emptyset$  then
4     return  $C$ ;
5    $i = \min(R)$ ;
6   List  $B_N^i = \text{ReadTopPowers}(i, N, M)$ ;
7   if  $B_N^i = \emptyset$  then
8     Set  $\tilde{R} = R \setminus \{i\}$ ;
9     return BuildTopBase( $M, \tilde{R}, N$ );
10  else
11    for  $j = \text{begin}(N)$  to  $j = \text{end}(N)$  do
12      if  $a_j^i = \max(B_N^i)$  then
13         $C[i] \leftarrow (a_j^i, i, j)$ ;
14        Set  $\tilde{R} = R \setminus \{i\}$ ;
15        Set  $\tilde{N} = N \setminus \{j\}$ ;
16        return BuildTopBase( $M, \tilde{R}, \tilde{N}$ );

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Below we describe the key steps.

- line 1. We initialize the Vector C to have 0 on all components and lenght r .
- line 6. We read the top powers from M on the line i . If we find any of these on the columns $j \in N$ then we include them in the list B_N^i .
- line 7. If there aren't any top powers on line i and the columns $j \in N$ from M then we skip to the next line in M .

Remark that a top base is not unique and it depends on the choice of the maximal top powers from each line, as the following example shows.

Example 3.2. Let $I = (x^{10}, y^{10}, z) \cap (x^{10}, y^2) \cap (x, z^4)$. Then we have that

$$M = \begin{pmatrix} \mathbf{10} & \mathbf{10} & 1 \\ \mathbf{10} & 2 & 0 \\ 1 & 0 & \mathbf{4} \end{pmatrix}$$

where the top powers are boldfaced.

We see that $\{a_1, a_2, a_3\} = \{10, 10, 4\}$ and we start the algorithm above to compute a top base. At the first step we can select $c_1 = a_1^1 = 10$.

$$\begin{pmatrix} \mathbf{10} & \mathbf{10} & 1 \\ \mathbf{10} & 2 & 0 \\ 1 & 0 & \mathbf{4} \end{pmatrix} \longrightarrow \begin{pmatrix} \cancel{\mathbf{10}} & \cancel{\mathbf{10}} & \cancel{1} \\ \cancel{\mathbf{10}} & 2 & 0 \\ \cancel{1} & 0 & \mathbf{4} \end{pmatrix}$$

Now in the second line we don't have any top powers, thus $c_2 = 0$ so we go to the third line from where we get $c_3 = 4$. So we obtained the top base $\{10, 0, 4\}$.

$$\begin{pmatrix} \cancel{\mathbf{10}} & \cancel{\mathbf{10}} & \cancel{1} \\ \cancel{\mathbf{10}} & 2 & 0 \\ \cancel{1} & 0 & \mathbf{4} \end{pmatrix} \longrightarrow \begin{pmatrix} \cancel{\mathbf{10}} & \cancel{\mathbf{10}} & \cancel{1} \\ \cancel{\mathbf{10}} & \cancel{2} & \emptyset \\ \cancel{1} & 0 & \mathbf{4} \end{pmatrix}$$

Now, if we choose $c_1 = a_2^1$ we obtain a different top base $\{10, 10, 4\}$.

$$\begin{pmatrix} \mathbf{10} & \mathbf{10} & 1 \\ \mathbf{10} & 2 & 0 \\ 1 & 0 & \mathbf{4} \end{pmatrix} \longrightarrow \begin{pmatrix} \cancel{\mathbf{10}} & \cancel{\mathbf{10}} & \cancel{1} \\ \mathbf{10} & \cancel{2} & 0 \\ 1 & \emptyset & \mathbf{4} \end{pmatrix} \longrightarrow \begin{pmatrix} \cancel{\mathbf{10}} & \cancel{\mathbf{10}} & \cancel{1} \\ \cancel{\mathbf{10}} & \cancel{2} & \emptyset \\ \cancel{1} & \emptyset & \mathbf{4} \end{pmatrix}$$

Definition 3.3. Consider the top base C from the above algorithm and set $c_i = C[i][1], i \in [r]$. In the following we denote

$$\overline{Q}_{i,j}^p = (X_{1,\min\{j,a_1^i\}}, \dots, X_{n,\min\{j,a_n^i\}}) \text{ for } j \in [c_i],$$

where $X_{i,0} = 0$ and if $c_i = 0$, then consider $\overline{Q}_{i,j} = (1)$ in intersections of ideals and $\overline{Q}_{i,j} = (0)$ in sums of ideals,

$$\overline{Q}_i^p = \bigcap_{j=1}^{c_i} \overline{Q}_{i,j}^p = (X_{1,1}, \dots, X_{n,1}) \cap (X_{1,2}, \dots, X_{n,2}) \cap \dots \cap (X_{1,\min\{c_i,a_1^i\}}, \dots, X_{n,\min\{c_i,a_n^i\}})$$

and

$$\overline{I}^p = \bigcap_{i=1}^r \overline{Q}_i^p.$$

Remark 3.4. Following Algorithm 1 we have that $\{c_1, \dots, c_r\} \setminus \{0\} \subset \{a_1, \dots, a_n\}$. Thus $\sum_{i=1}^r c_i \leq \sum_{k=1}^n a_k$, which implies that $\sum_{i=1}^r c_i \leq c + r$.

Definition 3.5. Set $A = \{\{j_1, \dots, j_{w+1}\} \text{ such that } \sqrt{\sum_{k=1}^{w+1} Q_{j_k}} = \sqrt{\sum_{j=1}^r Q_j}\}$, $w = \text{size } I\}$. We also use the notations $|A| = m$ and $A = \cup_{h=1}^m A_h$.

Remark 3.6. We may suppose (eventually after renumbering the ideals Q_i) that we always have $A_1 = \{1, 2, \dots, w+1\} \in A$, where $w = \text{size } I$.

Theorem 3.7. $\text{size } I^p \leq \text{size } I + c$.

Proof. From the definition of size we see that $\text{size } Q_i^p = \text{size } \overline{Q_i^p}$ and that $\text{size } I^p = \text{size } \overline{I^p}$, where

$$\overline{I^p} = \bigcap_{i=1}^r \overline{Q_i^p} = \bigcap_{\substack{1 \leq i \leq r \\ 1 \leq j \leq c_i}} \overline{Q_{i,j}^p},$$

therefore $\overline{I^p}$ has an irredundant decomposition composed of $D = \sum_{i=1}^r c_i$ irreducible monomial ideals.

According to Remark 3.6 we can assume that $\sqrt{Q_j} \subset \sqrt{\sum_{i=1}^{w+1} Q_i}$, for all $w+2 \leq j \leq r$

and set $a = r - (w+1)$. We see that $\sqrt{Q_j} \subset \sqrt{\sum_{i=1}^{w+1} Q_i}$, for all $w+2 \leq j \leq r$ implies that

$$\overline{Q_{j,1}^p} \subset \sum_{i=1}^{w+1} \overline{Q_{i,1}^p}, \text{ for all } w+2 \leq j \leq r.$$

Then

$$(3.1) \quad \text{size } I^p = \text{size } \overline{I^p} = \text{size} \left[\left(\bigcap_{\substack{1 \leq i \leq w+1 \\ 1 \leq j \leq c_i}} \overline{Q_{i,j}^p} \right) \cap \left(\bigcap_{\substack{w+2 \leq i \leq r \\ 2 \leq j \leq c_i}} \overline{Q_{i,j}^p} \right) \right].$$

Notice that the last term has $D - a$ intervals. Then $\text{size } I^p \leq D - a - 1 = D - r + w + 1 - 1 \leq c + w = \text{size } I + c$. \square

Remark 3.8. We see from equation 3.1 that $\text{size } I^p \leq \text{size } I + \sum_{\substack{i=1 \\ c_i > 0}}^r (c_i - 1) \leq \text{size } I + c$.

Thus, if $\{a_i : a_i > 1, i \in [n]\} \not\subseteq \{c_1, \dots, c_r\}$, we have that

$$\text{size } I^p \leq \text{size } I + \sum_{\substack{i=1 \\ c_i > 0}}^r (c_i - 1) < \text{size } I + \sum_{i=1}^n (a_i - 1) = \text{size } I + c.$$

Example 3.9. Let $I = (X_1^2, X_2^2) \cap (X_3^2, X_4^2) = Q_1 \cap Q_2 \subset \mathbb{K}[X_1, X_2, X_3, X_4]$ a monomial ideal. Then $\text{size } I^p = 3 < \text{size } I + c = 5$. Indeed we see that $\text{size } I = 1$ and that $c = 4$. Also note that

$$\text{size } I = \text{size } \overline{Q_{1,1}^p} \cap \overline{Q_{1,2}^p} \cap \overline{Q_{2,1}^p} \cap \overline{Q_{2,2}^p} = 3.$$

Let $I \subset S = \mathbb{K}[X_1, \dots, X_n]$ be a monomial ideal and let $I = \bigcap_{i=1}^r Q_i$ an irredundant decom-

position of I as an intersection of irreducible ideals, where the $Q_i = (X_1^{a_i^1}, \dots, X_n^{a_i^n}), a_j^i \in \mathbb{N}, j \in [n]$ are monomial ideals. Let Q_i be P_i -primary. For every $k \in [n]$, we set:

$$T_k = \{t : X_k^s \in G(Q_t), s \geq 2\};$$

$$L_k = \{l : X_k \in P_l\};$$

$$U_k = \{u : X_k \in Q_u, \max(B_{[n]}^u) > 1\}.$$

We may now formulate the main result of this paper, which fully describes the (particular) cases when the equality holds.

Theorem 3.10. *Let $I \subset S$ be a monomial ideal. Then $\text{size } I^p = \text{size } I + c$ if and only if $Q_i = \langle \{X_{i_1}^{a_{i_1}}, X_{i_2}, \dots, X_{i_s}\} \mid s \in [n], a_{i_1} \geq 1 \rangle$ for all $i \in [r]$ and, if $X_k \in P_i \cap P_j$, $1 \leq i < j \leq n$, $k \in [n]$, then one of the following is true:*

- (1) $T_k = \emptyset$ or
- (2) $T_k \neq \emptyset$ and
 - (A) $\forall t \in T_k, \{t, l\} \not\subseteq A_h, \forall l \in L_k \setminus \{t\}, \forall h \in [m]$ and
 - (B) if there exists $t \in T_k \cap A_h$ for some $h \in [m]$, then $U_k = \emptyset$.

Proof. " \Leftarrow "

(1) Let $Q_i = (X_i^{a_i}, X_{i_1}, \dots, X_{i_s})$, $i \in [r]$, $a_i > 1$, $t \in [n]$. If such a Q_i does not exist, then $c = 0$ and $I = I^p$. Thus we have that $X_i \notin P_j$ for all $j \in [r] \setminus \{i\}$, thus $i \in A_h$, for all $h \in [m]$. We may assume that Q_i , $i \in [s]$ have a top power ≥ 2 and that Q_i , $i > s$ are generated by variables, that is $\max(B_{[n]}^i) = 1$, for all $i > s$. Then the ideals in the second intersection from the last term in the equation 3.1 are generated by variables. Then we get that

$$\text{size } I^p = \sum_{i=1}^s c_i + w - s = \sum_{i=1}^n (a_i - 1) + w = \text{size } I + c,$$

because for all $j \in [a_i^i]$ and $i \in [s]$ we only have the ideal $(X_{i,j}, X_{i_1,1}, \dots, X_{i_s,1})$ to cover $X_{i,j}$.

(2) Let $\text{size } I = w$. Consider that Q_{w+2}, \dots, Q_s have top powers ≥ 2 and that Q_{s+1}, \dots, Q_r have top powers $= 1$. Then according to algorithm 1 we have that $c_i \leq 1$, for all $i > s$ and using equation 3.1 we see that for computing $\text{size } I^p$ we don't need the ideals Q_i , $i > s$. Moreover, we shall consider only the ideals Q_i , $w+1 < i \leq s$ with $c_i > 1$. Thus we may suppose that $c_i > 1$, $w+1 < i \leq s$.

If, for example, we have that $C[i] = (a_i^j, i, i)$, $i \geq w+2$, we show that in the sum of the other intervals we can cover only one variable from $\{X_{i,1}, \dots, X_{i,a_i^j}\}$ and that variable is $X_{i,1}$ from our choice in definition 3.3. Indeed, if we have $x_i^t \in G(Q_1)$, $t < a_i^j$, then c_1 can be at most 1, so we cover $X_{i,1}$. Condition (A) tells us that $x_i \notin P_j$, $1 < j \leq w+1$. Condition (B) tells us that if we have, for example, $C[1] = (a_1^d, 1, 1)$ and $C[i] = (a_i^j, i, i)$, $i \neq 1$, $a_i^j > 1$, then $x_1 \notin G(Q_i)$, so that we can not cover $X_{1,1}$ in the sum of the other intervals, that is $\overline{Q_{i,l}^p}$, $2 \leq l \leq a_i^j$.

It follows that

$$\text{size } I^p = \sum_{i=1}^{w+1} c_i + \sum_{i=w+2}^s (c_i - 1) - 1 = \text{size } I + \sum_{i=1}^{w+1} (c_i - 1) + \sum_{i=w+2}^s (c_i - 1) = \text{size } I + c.$$

" \Rightarrow "

Assume that $\text{size } I^p = \text{size } I + c$. Then Remark 3.8 gives us the following inclusion $\{a_i : a_i > 1, i \in [n]\} \subset \{c_1, \dots, c_r\}$. Let $\text{size } I = w$.

First suppose that $X_1^{a_1^1}, X_2^{a_2^1} \in G(Q_1)$, $a_1^j > 1$, $j \in [2]$. If $c_1 > a_1^1$, then using Remark 3.8 we get that there exists $1 < j \leq r$ such that $c_j = a_1^j = a_1$. Then from equation 3.1 we can skip the ideal $\overline{Q_{j,2}^p}$ because we have $X_{1,2} \in \overline{Q_{1,2}^p}$ and we find all the other variables $X_{p,2}$, $2 \leq p \leq n$ in $\overline{Q_{w,2}^p}$, where $c_w = a_p$. We will call this procedure *the absence of variable* $X_{1,2}$. Thus we get that $\text{size } I^p < \text{size } I + c$.

Now, if $c_1 = a_1^1 = a_1$ let $2 \leq j \leq r$ such that $c_j = a_2$. Then again, using *the absence of variable* $X_{2,2}$, we get that $\text{size } I^p < \text{size } I + c$. Thus we see that $a_i^j = 1$ except for at most one $i \in [n]$, for all $j \in [r]$.

Suppose that $\{1, 2\} \subset A_1$, $X_1^{a_1^1} \in G(Q_1)$, $X_1^{a_1^2} \in G(Q_2)$, $a_1^1 > 1$, $a_1^2 \geq 1$, $a_1^1 \geq a_1^2$. If $c_1 = a_1^1$, then

$$\begin{aligned} \text{size } I^p &\leq \text{size} \left[\left(\bigcap_{\substack{1 \leq i \leq w+1 \\ 1 \leq j \leq c_i}} \overline{Q_{i,j}^p} \right) \cap \left(\bigcap_{\substack{w+2 \leq i \leq r \\ 2 \leq j \leq c_i}} \overline{Q_{i,j}^p} \right) \right] \\ &\leq \text{size} \left[\left(\bigcap_{\substack{2 \leq i \leq w+1 \\ 1 \leq j \leq c_i}} \overline{Q_{i,j}^p} \right) \cap \left(\bigcap_{\substack{w+2 \leq i \leq r \\ 2 \leq j \leq c_i}} \overline{Q_{i,j}^p} \cap \overline{Q_{1,2}^p} \cdots \cap \overline{Q_{1,a_1^1}^p} \right) \right] \\ &\leq \sum_{i=2}^{w+1} c_i + \sum_{i=w+2}^r (c_i - 1) + c_1 - 1 \\ &= \text{size } I - 1 + c. \end{aligned}$$

We skipped the ideal $\overline{Q_{1,1}^p}$ because $\overline{Q_{1,1}^p} \subset \overline{Q_{1,2}^p} + \sum_{j=2}^{w+1} \overline{Q_{j,1}^p}$.

If $c_1 > a_1^1$, then there exists $2 \leq j \leq r$ such that $c_j = a_1^j = a_1 > 1$ and in equation 3.1 we can skip over the ideal $\overline{Q_{j,1}^p}$ because $\overline{Q_{j,1}^p} \subset \overline{Q_{j,2}^p} + \sum_{\substack{i=1 \\ i \neq j}}^{w+1} \overline{Q_{i,1}^p}$, thus $\text{size } I^p \leq \text{size } I + c - 1$.

Now suppose that $1 \in A_1$, $X_1^{a_1^1} \in G(Q_1)$, $a_1^1 > 1$, $X_1, X_2^{a_2^u} \in G(Q_u)$ and $c_u = a_2^u > 1$. As we have seen above, we may assume that $c_1 = a_1^1$. Then we have $\overline{Q_{1,1}^p} \subset \sum_{j=2}^{w+1} \overline{Q_{j,1}^p} + \overline{Q_{u,2}^p}$, thus we get $\text{size } I^p \leq \text{size } I + c - 1$. \square

Example 3.11. Consider the monomial ideal $I = (X_1^2, X_2) \cap (X_2, X_3) \cap (X_3, X_4) \cap (X_2, X_4) = \bigcap_{i=1}^4 Q_i \subset \mathbb{K}[X_1, X_2, X_3, X_4]$. Then $A = \{\{1, 2, 3\}, \{1, 2, 4\}\}$ and $c = 1$, thus $\text{size } I = 2$. Using Theorem 3.10 we get that $\text{size } I^p = \text{size } I + c$. Indeed, $\text{size } I^p = \text{size}(X_{1,1}, X_{2,1}) \cap (X_{1,2}, X_{2,1}) \cap (X_{2,1}, X_{3,1}) \cap (X_{3,1}, X_{4,1}) \cap (X_{2,1}, X_{4,1})$.

Example 3.12. Let $I = (X_1^2, X_2) \cap (X_1, X_3) \cap (X_2, X_3) = \bigcap_{i=1}^3 Q_i \subset \mathbb{K}[X_1, X_2, X_3]$. Then $A = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $c = 1$, thus $\text{size } I = 1$. We have that $\text{size } I^p = \text{size}(X_{1,2}, X_{2,1}) \cap (X_{1,1}, X_{3,1}) = \text{size } \overline{Q_{1,2}^p} \cap \overline{Q_{2,1}^p} = 1$, thus $\text{size } I^p < \text{size } I + c$. We see that I does not respect condition (2)(A) from Theorem 3.10.

Example 3.13. Let $I = (X_1^2, X_2) \cap (X_3, X_4) \cap (X_1, X_4^2) = \bigcap_{i=1}^4 Q_i \subset \mathbb{K}[X_1, X_2, X_3, X_4]$. Then $A = \{\{1, 2\}\}$ and $c = 2$ thus $\text{size } I = 1$. We have that $\text{size } I^p = \text{size}(X_{1,2}, X_{2,1}) \cap (X_{3,1}, X_{4,1}) \cap (X_{1,1}, X_{4,2}) = \text{size } \bar{Q}_{1,2}^p \cap \bar{Q}_2^p \cap \bar{Q}_{3,2}^p = 2$, thus $\text{size } I^p < \text{size } I + c$. We see here that X_4 respects all the conditions from Theorem 3.10, but X_1 does not respect the condition (2)(B).

Example 3.14. Let $I = (X_1^{k+1}, X_2^k) \cap (X_1, X_2^{k+1}) \subset \mathbb{K}[X_1, X_2]$. Then $\text{size } I^p = \text{size } I + c - k$. Indeed, we have $\text{size } I = 0, c = 2k$ and $\text{size } I^p = \text{size}(X_{1,2}, X_{2,1}) \cap \dots \cap (X_{1,k+1}, X_{2,k}) \cap (X_{1,1}, X_{2,k+1})$. We see that each variable appears only once, thus the size can not be smaller.

As an application of our main result, by using Theorem 3.10 and Theorem 2.5, we easily deduce the following extension of Theorem 2.6.

Corollary 3.15. *Let I be a monomial ideal of S such that, either I is squarefree, or I is as described in Theorem 3.10. Then*

- (1) $\text{sdepth } I \geq \text{size } I + 1$;
- (2) $\text{sdepth } S/I \geq \text{size } I$.

4. THE BEHAVIOR OF SIZE UNDER GENERIC DEFORMATIONS

The notion of *deformation* of a monomial ideal was introduced by Bayer et al. [2] and further developed in Miller et al. [13].

Definition 4.1. (1) Let $\mathcal{M} = \{m_1, \dots, m_r\} \subset S$ be a set of monomials. For $1 \leq i \leq r$ let $a^i = (a_1^i, \dots, a_n^i) \in \mathbb{N}^n$ denote the exponent vector of m_i . A *deformation* of \mathcal{M} is a set of vectors $\varepsilon_i = (\varepsilon_1^i, \dots, \varepsilon_n^i) \in \mathbb{N}^n$ for $1 \leq i \leq r$ subject to the following conditions:

$$(4.1) \quad a_j^i > a_j^k \implies a_j^i + \varepsilon_j^i > a_j^k + \varepsilon_j^k \quad \text{and} \quad a_k^i = 0 \implies \varepsilon_k^i = 0.$$

(2) Let $I \subset S$ be a monomial ideal with generating set G_I . A *deformation* of I is a deformation of G_I . We set $I_\varepsilon := (g \cdot \mathbf{x}^{\varepsilon_g} : g \in G_I)$ to be the ideal generated by the deformed generators.

The most important deformations are the generic deformations. Let us recall the definition from [13].

Definition 4.2. (1) A monomial $m \in S$ is said to *strictly divide* another monomial $m' \in S$ if $m \mid \frac{m'}{X_i}$ for each variable X_i dividing m' .

(2) A monomial ideal $I \subset S$ is called *generic* if for any two minimal generators m, m' of I having the same degree in some variable, there exists a third minimal generator m'' that strictly divides $\text{lcm}(m, m')$.

(3) A deformation of a monomial ideal I is called *generic* if the deformed ideal I_ε is generic.

Definition 4.3. [14] A monomial ideal is considered generic if for any $m_1, m_2 \in \text{Min}(I)$ and any variable X_i if $\deg_{X_i}(m_1) = \deg_{X_i}(m_2)$, then $\deg_{X_i}(m_1) = 0$. That is, no minimal generator of I has the same exponent for any variable.

Proposition 4.4. *Let $I \subset S = \mathbb{K}[X_1, \dots, X_n]$ be a monomial ideal and let $I = \bigcap_{i=1}^r Q_i$ an irredundant decomposition of I as an intersection of irreducible ideals, where $\sqrt{Q_i} = P_i$. Let I_ε be the generic deformation of I . Then $\text{size } I \geq \text{size } I_\varepsilon$.*

Proof. We follow the proof of [14] Theorem 6, except we consider any generic transformation, not just the generic ones. By construction, we have an unique map φ from the individual powers of variables amongst the generators of I to individual powers of variables amongst the generators of I_ε . Let $I_\varepsilon = \bigcap_{i=1}^r Q_i$ be the unique irredundant irreducible

primary decomposition of I_ε , then by [14] Theorem 6 we have that $I = \bigcap_{i=1}^r \varphi(Q_i)$ is an irreducible primary decomposition of I . The decomposition may not be irredundant.

If we consider $\text{size } I_\varepsilon = t < r$ with $\sqrt{\sum_{j=1}^t Q_{i_j}} = \sqrt{\sum_{i=1}^r Q_i}$, then we also have

$$\sqrt{\sum_{j=1}^t \varphi(Q_{i_j})} = \sqrt{\sum_{i=1}^r \varphi(Q_i)} = \sqrt{\sum_{i=1}^r Q_i}.$$

Thus we get the inequality $\text{size } I \geq \text{size } I_\varepsilon$, because the decomposition for I may be redundant. \square

Example 4.5. Let $I = (xyt, xyw, xtw, yzt, yzw, ztw) \subset K[z, y, z, t, w]$ and let ε_1 be a deformation and ε_2 a generic deformation such that $I_{\varepsilon_1} = (xyt, xyw, xt^3w, yzt^2, yzw, ztw)$ and $I_{\varepsilon_2} = (x^3y^4t, x^2y^2w, xt^3w^3, y^3zt^2, yz^2w^2, z^3tw^4)$. Then we have the following unique irreducible primary decompositions:

$$\begin{aligned} I &= (x, z) \cap (y, t) \cap (y, w) \cap (t, w); \\ I_{\varepsilon_1} &= (x, z) \cap (y, t) \cap (y, w) \cap (t, w) \cap (x, t^2, w) \cap (y, z, t^3); \\ I_{\varepsilon_2} &= (x, z) \cap (y, t) \cap (y^3, y^2w, yw^2, w^3) \cap (t, w) \cap (y^4, y^3t^2, y^2w, yw^2, t^3, w^4) \cap \\ &\quad (x^3, x^2w, t^2, w^2) \cap (y^4, y^3z, y^2w, z^2, w^3) \cap (x^3, x^2w, z, w^3) \cap (x^2, xw^3, y^3, yw^2, w^4) \cap \\ &\quad (y^2, yz^2, z^3, t^3) \cap (x^2, z^2, zt^2, t^3) \cap (x, y^3, yz^2, z^3) \cap (x^3, x^2y^2w, x^2z^2, y^3, yz^2w^2, z^3, w^3) \cap \\ &\quad (x^3, x^2y^2, y^3, yz^2, z^3, t^3). \end{aligned}$$

We have that $2 = \text{size } I > \text{size } I_{\varepsilon_1} = \text{size } I_{\varepsilon_2} = 1$, thus the inequality from the above proposition may be strict.

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